



12.7 Solving a system of homogenous linear ordinary differential equations





Introduction

- In this topic, we will
 - Discuss how the differential operator is linear
 - See that the differential operator has eigenvalues and eigenfunctions
 - See how these eigenfunctions are used to find solutions to a system of ordinary differential equations
 - See that this results in unknowns, which can be fixed based on initial conditions
 - Remind ourselves that this is supplementary information
 - This will be taught in detail in second year





Eigenvectors of the differential operator

- Recall that for $\mathbf{u}$ to be an eigenvector for a matrix A , we must have that $A\mathbf{u} = \lambda\mathbf{u}$
 - Recall that the defining property of a linear operator is
$$A(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha A\mathbf{u} + \beta A\mathbf{v}$$
- For differentiable functions, is the differential operator linear?
$$\frac{d}{dt}(\alpha x(t) + \beta y(t)) = \alpha \frac{d}{dt} x(t) + \beta \frac{d}{dt} y(t)$$
 - Yes it is!





Eigenvectors of the differential operator

- Referring back to $A\mathbf{u} = \lambda\mathbf{u}$,
the differential operator may also have eigenvectors
 - Are there any functions such that

$$\frac{d}{dt}y(t) = \lambda y(t) ?$$

- You should recognize the following:

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$$

- The derivative of this exponential is λ times this same exponential
- Every real or complex number is an eigenvalue of this operator





Eigenvectors of the differential operator

- Can any polynomial $p(t)$ be an eigenfunction?

$$\frac{d}{dt} p(t) = \lambda p(t)$$

- If the degree of p is $n > 0$,
then the degree of the derivative must be $n - 1$
 - Therefore, no non-constant polynomial can be an eigenfunction
- What if we have a constant polynomial?

$$\frac{d}{dt} c = 0$$

- Thus, the constant functions are in the null space of the differential operator
 - The derivative applied to the constant function produces the zero function—the zero vector for functions





Differential equations

- Engineering deals with forces in the real world
 - Forces are described by derivatives and second derivatives
- In some cases, you may have two or more systems, where the response of one system depends on the others
 - For example, in an electric circuit, you may have capacitors, inductors and resistors
 - The response of the inductors will change depending on the charge of any connected capacitors
 - For example, given a nano-surface, you may have different points of pressure, and that pressure may cause that surface to deform
 - How do you determine the deformation?





Differential equations

- In some cases, such systems may be written as a system of homogenous linear ordinary differential equations:

$$\frac{d}{dt}x(t) = a_{1,1}x(t) + a_{1,2}y(t) + a_{1,3}z(t)$$

$$\frac{d}{dt}y(t) = a_{2,1}x(t) + a_{2,2}y(t) + a_{2,3}z(t)$$

$$\frac{d}{dt}z(t) = a_{3,1}x(t) + a_{3,2}y(t) + a_{3,3}z(t)$$





Differential equations

- First, note that we can rewrite these ordinary differential equations as follows:

$$\frac{d}{dt}x(t) = a_{1,1}x(t) + a_{1,2}y(t) + a_{1,3}z(t)$$

$$\frac{d}{dt}y(t) = a_{2,1}x(t) + a_{2,2}y(t) + a_{2,3}z(t)$$

$$\frac{d}{dt}z(t) = a_{3,1}x(t) + a_{3,2}y(t) + a_{3,3}z(t)$$

$$\begin{pmatrix} \frac{d}{dt}x(t) \\ \frac{d}{dt}y(t) \\ \frac{d}{dt}z(t) \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$





Differential equations

- Without proof,
we can find functions that simultaneously satisfy all three of
these conditions as follows:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$$

- Find the eigenvalues and corresponding eigenvectors

$$\lambda_1, \mathbf{u}_1 \quad \lambda_2, \mathbf{u}_2 \quad \lambda_3, \mathbf{u}_3$$





Differential equations

- The solution therefore is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 \mathbf{u}_1 e^{\lambda_1 t} + c_2 \mathbf{u}_2 e^{\lambda_2 t} + c_3 \mathbf{u}_3 e^{\lambda_3 t}$$

- The constants c_1 , c_2 and c_3 are free variables that are fixed by initial conditions, e.g.,

$$x(0) = x_0$$

$$y(0) = y_0$$

$$z(0) = z_0$$





Differential equations

- Now, $e^{\lambda \cdot 0} = 1$, so

$$\begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = c_1 \mathbf{u}_1 e^0 + c_2 \mathbf{u}_2 e^0 + c_3 \mathbf{u}_3 e^0 = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3$$

- This is nothing more than a system of linear equations:

$$c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + c_3 \mathbf{u}_3 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$





Solving a system of two initial-value problems

- For example,

$$\frac{d}{dt}x(t) = -2x(t) + 2y(t)$$

$$\frac{d}{dt}y(t) = x(t) - 3y(t)$$





Solving a system of two initial-value problems

- The right-hand side defines a matrix:

$$\begin{aligned}\frac{d}{dt}x(t) &= -2x(t) + 2y(t) \\ \frac{d}{dt}y(t) &= x(t) - 3y(t)\end{aligned}\quad A = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix} \quad \begin{pmatrix} \frac{d}{dt}x(t) \\ \frac{d}{dt}y(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

- The matrix A has these eigenvalues and eigenvectors

$$-4, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad -1, \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$





Solving a system of two initial-value problems

- With these eigenvalue-eigenvector pairs,
we claim the solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-4t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}$$

- If you prefer:

$$x(t) = -c_1 e^{-4t} + 2c_2 e^{-t}$$

$$y(t) = c_1 e^{-4t} + c_2 e^{-t}$$





Solving a system of two initial-value problems

- Suppose we know that at time $t = 0$,
both functions have these initial values:

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix} \quad \begin{matrix} x(0) = 7 \\ y(0) = 8 \end{matrix}$$

- In this case, we have:

$$\begin{matrix} x(0) = -c_1 + 2c_2 = 7 \\ y(0) = c_1 + c_2 = 8 \end{matrix} \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & 2 & 7 \\ 1 & 1 & 8 \end{array} \right) \sim \left(\begin{array}{cc|c} -1 & 2 & 7 \\ 0 & 3 & 15 \end{array} \right) \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$





Solving a system of two initial-value problems

- We substitute this solution into our functions:

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \begin{aligned} x(t) &= -c_1 e^{-4t} + 2c_2 e^{-t} \\ y(t) &= c_1 e^{-4t} + c_2 e^{-t} \end{aligned}$$

- We therefore have the following:

$$\begin{aligned} x(t) &= -3e^{-4t} + 10e^{-t} \\ y(t) &= 3e^{-4t} + 5e^{-t} \end{aligned}$$

- Note these functions satisfy both:

$$\begin{aligned} \frac{d}{dt} x(t) &= -2x(t) + 2y(t) & x(0) &= 7 \\ \frac{d}{dt} y(t) &= x(t) - 3y(t) & y(0) &= 8 \end{aligned}$$



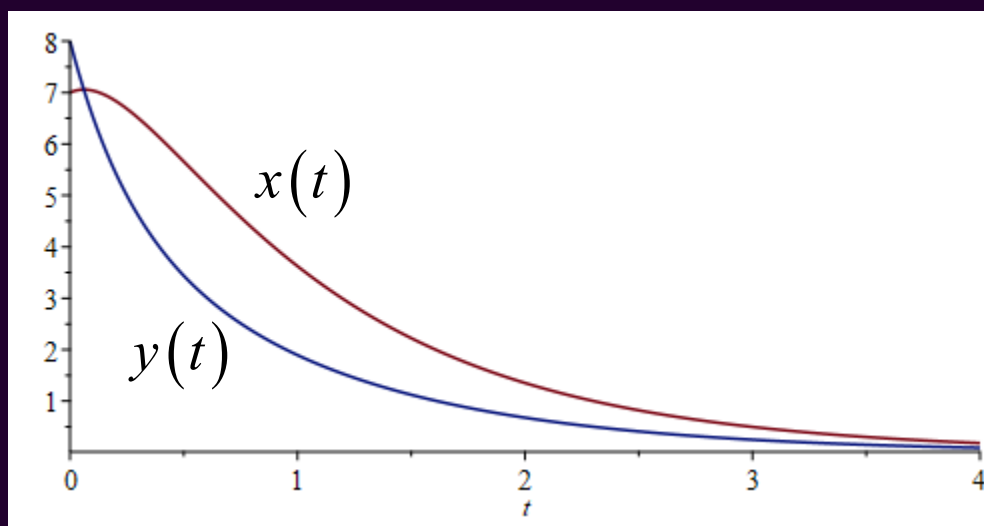


Solving a system of two initial-value problems

- Here are these two functions:

$$x(t) = -3e^{-4t} + 10e^{-t}$$

$$y(t) = 3e^{-4t} + 5e^{-t}$$





Solving a system of four initial-value problems

- For example,

$$\frac{d}{dt}w(t) = 2x(t) + y(t) - 2z(t)$$

$$\frac{d}{dt}x(t) = 2w(t) + 2x(t) + 2y(t) - 2z(t)$$

$$\frac{d}{dt}y(t) = w(t) + 2x(t) - 2z(t)$$

$$\frac{d}{dt}z(t) = -2w(t) - 2x(t) - 2y(t) - z(t)$$





Solving a system of four initial-value problems

- As before, this system is described by this matrix:

$$\frac{d}{dt}w(t) = 2x(t) + y(t) - 2z(t)$$

$$\frac{d}{dt}x(t) = 2w(t) + 2x(t) + 2y(t) - 2z(t)$$

$$\frac{d}{dt}y(t) = w(t) + 2x(t) - 2z(t)$$

$$\frac{d}{dt}z(t) = -2w(t) - 2x(t) - 2y(t) - z(t)$$

$$A = \begin{pmatrix} 0 & 2 & 1 & -2 \\ 2 & 2 & 2 & -2 \\ 1 & 2 & 0 & -2 \\ -2 & -2 & -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \dot{w}(t) \\ \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = A \begin{pmatrix} w(t) \\ x(t) \\ y(t) \\ z(t) \end{pmatrix}$$





Solving a system of four initial-value problems

- The matrix A has these eigenvalues and eigenvectors

$$\begin{aligned} \dot{w}(t) &= 2x(t) + y(t) - 2z(t) \\ \dot{x}(t) &= 2w(t) + 2x(t) + 2y(t) - 2z(t) \\ \dot{y}(t) &= w(t) + 2x(t) - 2z(t) \\ \dot{z}(t) &= -2w(t) - 2x(t) - 2y(t) - z(t) \end{aligned} \quad A = \begin{pmatrix} 0 & 2 & 1 & -2 \\ 2 & 2 & 2 & -2 \\ 1 & 2 & 0 & -2 \\ -2 & -2 & -2 & -1 \end{pmatrix}$$

$$6, \begin{pmatrix} 2 \\ 3 \\ 2 \\ -2 \end{pmatrix} \quad -1, \begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix} \quad -1, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad -3, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$





Solving a system of four initial-value problems

- With these eigenvalue-eigenvector pairs,
we claim the solution is

$$\begin{pmatrix} w(t) \\ x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3 \\ 2 \\ -2 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{-t} + c_4 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} e^{-3t}$$

- If you prefer:

$$w(t) = 2c_1 e^{6t} + c_2 e^{-t} + c_3 e^{-t} + c_4 e^{-3t}$$

$$x(t) = 3c_1 e^{6t} - 2c_2 e^{-t}$$

$$y(t) = 2c_1 e^{6t} + c_2 e^{-t} - c_3 e^{-t} + c_4 e^{-3t}$$

$$z(t) = -2c_1 e^{6t} - c_2 e^{-t} + 2c_4 e^{-3t}$$





Solving a system of four initial-value problems

- Suppose we know that at time $t = 0$,
we have the following initial conditions:

$$\begin{pmatrix} w(0) \\ x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} -7.5 \\ 8.5 \\ -0.5 \\ 20.5 \end{pmatrix} \quad \begin{aligned} w(0) &= -7.5 \\ x(0) &= 8.5 \\ y(0) &= -0.5 \\ z(0) &= 20.5 \end{aligned}$$

- In this case, we have:

$$\begin{pmatrix} w(0) \\ x(0) \\ y(0) \\ z(0) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3 \\ 2 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -7.5 \\ 8.5 \\ -0.5 \\ 20.5 \end{pmatrix}$$





Solving a system of four initial-value problems

- This defines the following system of linear equations:

$$\begin{pmatrix} 2 & 1 & 1 & 1 & | & -7.5 \\ 3 & -2 & 0 & 0 & | & 8.5 \\ 2 & 1 & -1 & 1 & | & -0.5 \\ -2 & -1 & 0 & 2 & | & 20.5 \end{pmatrix}$$

- The solution is:

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -1.5 \\ -6.5 \\ -3.5 \\ 5.5 \end{pmatrix}$$





Solving a system of four initial-value problems

- Substituting these into our general solution:

$$\begin{pmatrix} w(t) \\ x(t) \\ y(t) \\ z(t) \end{pmatrix} = -1.5 \begin{pmatrix} 2 \\ 3 \\ 2 \\ -2 \end{pmatrix} e^{6t} - 6.5 \begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix} e^{-t} - 3.5 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{-t} + 5.5 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} e^{-3t}$$

– If you prefer:

$$w(t) = -3e^{6t} - 6.5e^{-t} - 3.5e^{-t} + 5.5e^{-3t} = -3e^{6t} - 10e^{-t} + 5.5e^{-3t}$$

$$x(t) = -4.5e^{6t} + 13e^{-t} = -4.5e^{6t} + 13e^{-t}$$

$$y(t) = -3e^{6t} - 6.5e^{-t} + 3.5e^{-t} + 5.5e^{-3t} = -3e^{6t} - 3e^{-t} + 5.5e^{-3t}$$

$$z(t) = 3e^{6t} + 6.5e^{-t} + 11e^{-3t} = 3e^{6t} + 6.5e^{-t} + 11e^{-3t}$$





Solving a system of four initial-value problems

- With this solution:
$$w(t) = -3e^{6t} - 10e^{-t} + 5.5e^{-3t}$$
$$x(t) = -4.5e^{6t} + 13e^{-t}$$
$$y(t) = -3e^{6t} - 3e^{-t} + 5.5e^{-3t}$$
$$z(t) = 3e^{6t} + 6.5e^{-t} + 11e^{-3t}$$

we note that these functions satisfy all these conditions:

$$\frac{d}{dt} w(t) = 2x(t) + y(t) - 2z(t)$$

$$w(0) = -7.5$$

$$\frac{d}{dt} x(t) = 2w(t) + 2x(t) + 2y(t) - 2z(t)$$

$$x(0) = 8.5$$

$$\frac{d}{dt} y(t) = w(t) + 2x(t) - 2z(t)$$

$$y(0) = -0.5$$

$$\frac{d}{dt} z(t) = -2w(t) - 2x(t) - 2y(t) - z(t)$$

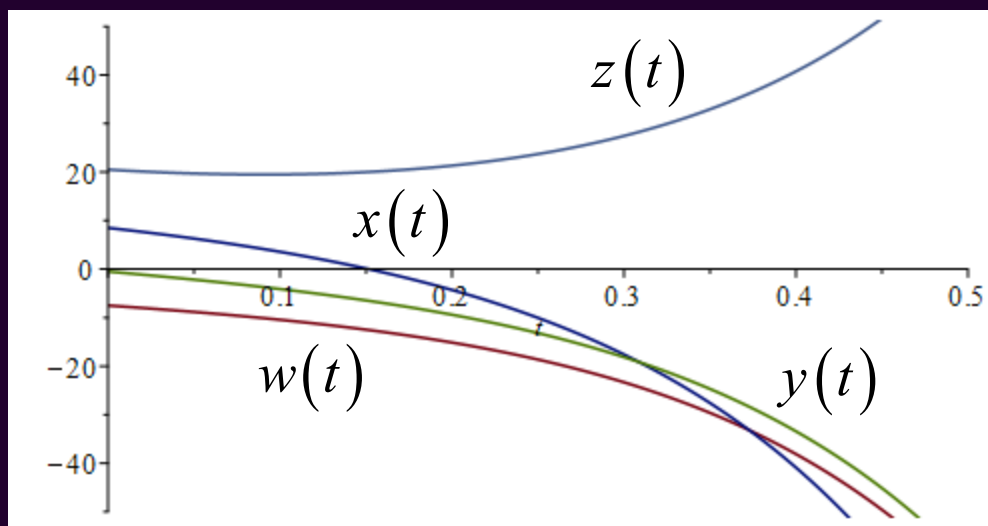
$$z(0) = 20.5$$





Solving a system of four initial-value problems

- With this solution:
$$w(t) = -3e^{6t} - 10e^{-t} + 5.5e^{-3t}$$
$$x(t) = -4.5e^{6t} + 13e^{-t}$$
$$y(t) = -3e^{6t} - 3e^{-t} + 5.5e^{-3t}$$
$$z(t) = 3e^{6t} + 6.5e^{-t} + 11e^{-3t}$$
 - Note these functions satisfy all these conditions:





Summary

- Following this topic, you now
 - Are aware that the eigenfunctions of the differential operator are the exponential functions
 - Understand that eigenvalues and eigenvectors can be used to find solutions to systems of initial-value problems
 - Know you are not required to understand the details for this course, but you are aware of how the differential equations are translated into a matrix
 - Are aware that in reality, you will not need to find the eigenvalues and eigenvectors explicitly, but rather, you would use software to find these





References

- [1] https://en.wikipedia.org/wiki/Linear_differential_equation
- [2] https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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